Numerical simulation of blood flow

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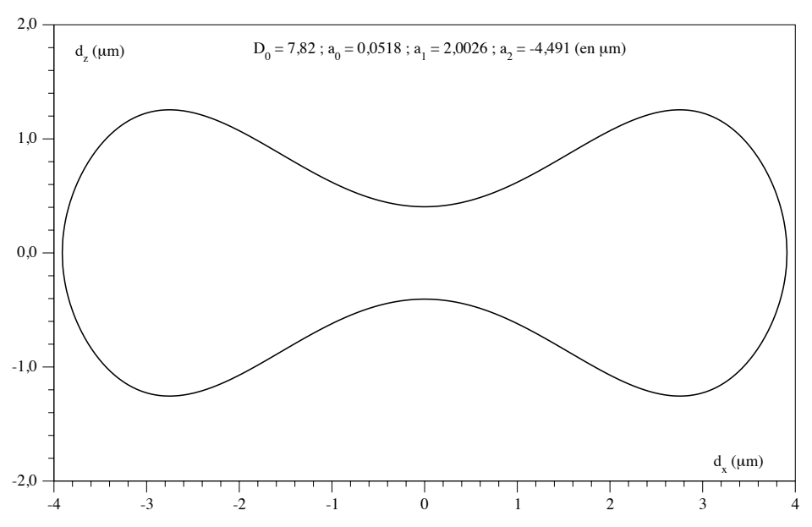
Motivation and goals

- Study of the behavior of human red blood cells
  - Physical modeling of the membrane's deformability due to its viscoelasticity
  - Mathematical modeling of the thin membrane
  - Numerical modeling of interfaces between different fluids
  - Numerical simulations
- Our approach
  - Membrane treated as a non-Newtonian viscoelastic fluid of Giesekus
  - Asymptotic modeling of the membrane as the thickness tends to 0
  - Extension of NXFEM method for interfaces to:
    - Nonconforming finite elements for elliptic and Stokes equations
    - Non-standard transmission conditions
  - Implementation in Concha library

1. Physical modeling of human red blood cells

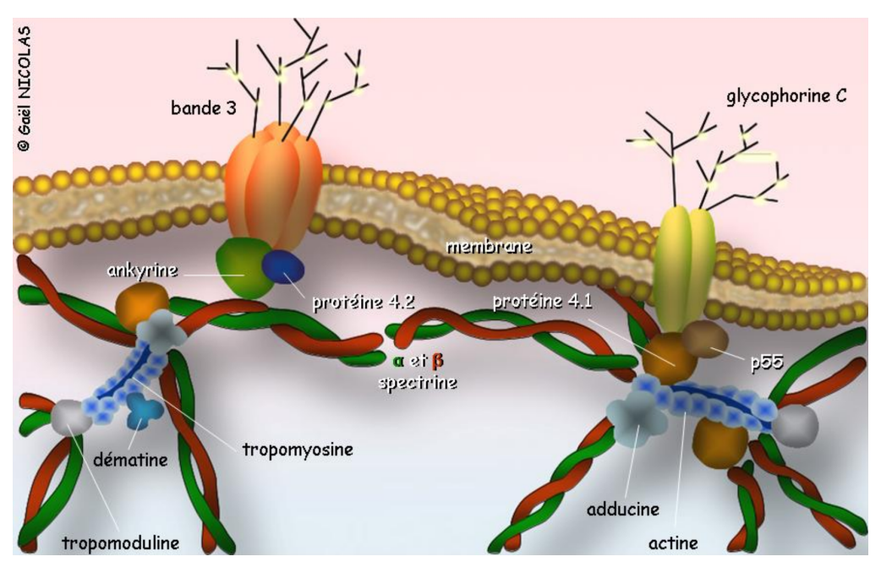
- Dimensions:
  - Cell's diameter $D_0 = 7{,}82\ \mu$m
  - Membrane's thickness $e_0 = 7{,}5$ nm



- Equation of the surface at equilibrium (Fedosov - Evans):

$$z = \pm D_0 \sqrt{1 - 4\frac{x^2+y^2}{D_0^2}} \left(a_0 + a_1 \frac{x^2+y^2}{D_0^2} + a_2 \frac{\left(x^2+y^2\right)^2}{D_0^4} \right)$$

- Membrane: composite material
  (lipid bilayer+spectrin elastic network)
  $\rightsquigarrow$ classically, the membrane is a
  hyper-elastic solid



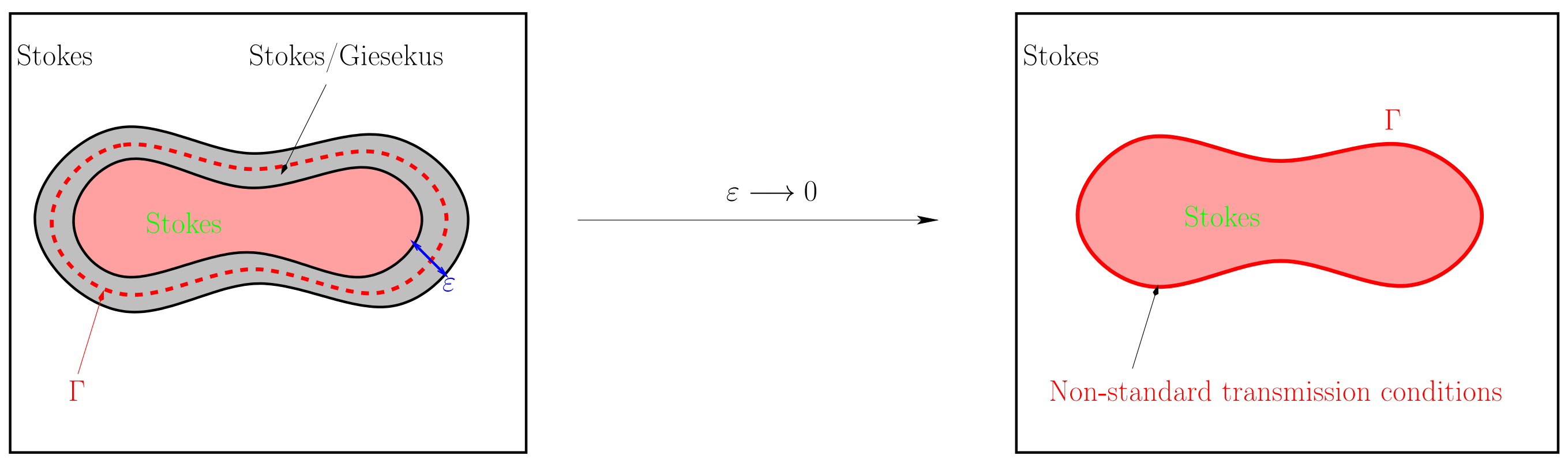
- Membrane: viscoelastic properties $\rightsquigarrow$ in our approach, it is a viscoelastic non-Newtonian liquid
- Realistic constitutive law of Giesekus:

$$\lambda\left(\frac{\partial}{\partial t}\tau + u\cdot\nabla\tau - \tau(\nabla u)^t - \nabla u\tau\right) + \tau + \frac{\lambda}{2\eta}\tau\cdot\tau = \eta\left((\nabla u)^t + \nabla u\right)$$

u the velocity, τ the viscous stress tensor, λ the relaxation time, η the viscosity

2. Mathematical modeling of the small thickness

- $\varepsilon := \frac{e_0}{D_0} \approx 10^{-3}$ $\rightsquigarrow$ problem to mesh the thin membrane !
- We consider the limit weak problem as ε tends to 0



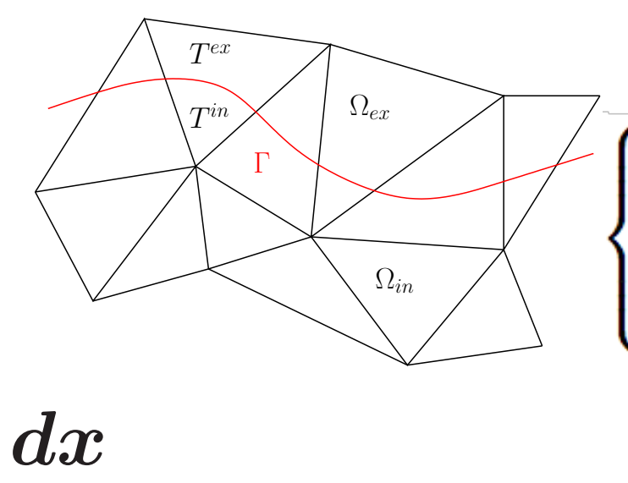
Initial model problem

Limit model problem

- Mathematical analysis in the case of:
  - Rectilinear interface
  - Stokes equations in the membrane
- Extension to smooth curved interface and to Giesekus law

3.1. Numerical modeling of interfaces: NXFEM

- Nitsche's Extended Finite Element Method (NXFEM):
  - introduced for conforming approx. of elliptic problems (Hansbo & Hansbo '02)
  - designed to take into account discontinuities on non-aligned meshes by:
    - standard FE spaces enriched on cut cells (doubled d.o.f. : $V_h = V_h^{in} \times V_h^{ex}$)
    - interface conditions treated by Nitsche's method (additional terms in the weak form)



$$\begin{cases} -\mathrm{div}\,(\eta\nabla u) = f & \text{in } \Omega^{in}\cup\Omega^{ex} \\ u = 0 & \text{on } \partial\Omega \\ [u] = 0 & \text{on } \Gamma \\ [\eta\nabla_n u] = g & \text{on } \Gamma \end{cases}$$

$$\begin{aligned} A_h(u_h, v_h) = & \sum_{T\in\mathcal{T}_h} \int_T \eta\nabla u_h\cdot\nabla v_h\,dx & \rightsquigarrow \text{Classical term} \\ & - \int_\Gamma \{\eta\nabla_n u_h\}[v_h]\,ds - \int_\Gamma \{\eta\nabla_n v_h\}[u_h]\,ds & \rightsquigarrow \text{Symmetrization} \\ & + \gamma \sum_{T\in\mathcal{T}_h^\Gamma} \frac{\eta_{in}\eta_{ex}|T|}{\eta_{ex}|T^{in}| + \eta_{in}|T^{ex}|} \int_{\Gamma_T} [u_h][v_h]\,ds & \rightsquigarrow \text{Stabilization} \end{aligned}$$

with γ a stabilization parameter

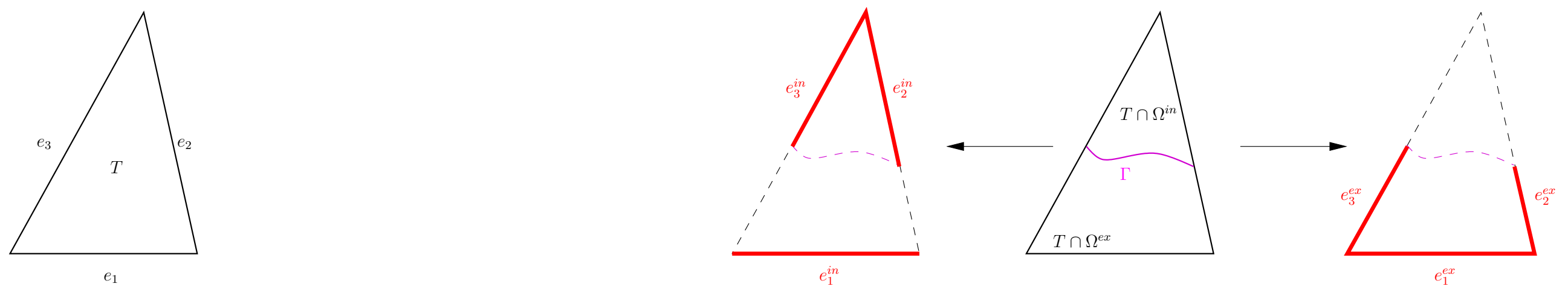
3.2. Numerical modeling of interfaces: our contributions

- Development of two variants of NXFEM method for:
  - Crouzeix-Raviart nonconforming finite elements
  - Elliptic and Stokes equations
- Main difficulties:
  - Nonconforming approximations: estimate the consistency error on cut edges
  - Stokes equations: inf-sup condition
- Proposed solutions

① Modification of the classical basis functions on the cut triangles

$$\varphi_i \in P_1(T),\ \frac{1}{|e_j|}\int_{e_j}\varphi_i ds = \delta_{ij} \qquad \tilde\varphi_i^k \in P_1(T),\ \frac{1}{\left|e_j^k\right|}\int_{e_j^k}\tilde\varphi_i^k\,ds = \delta_{ij}$$

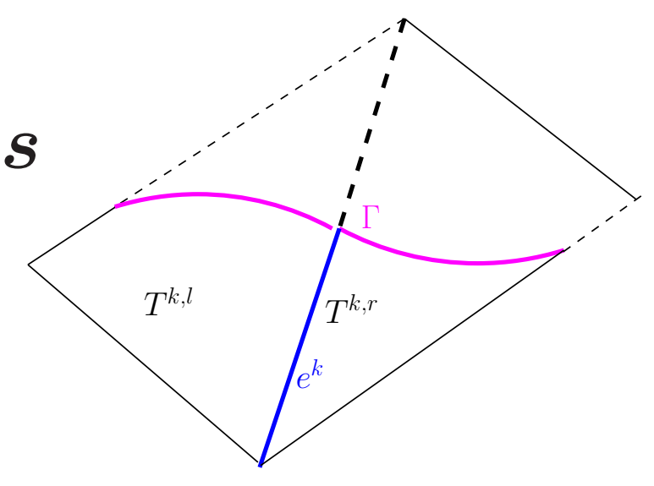
$$k = in,\ ex, \quad 1 \le i, j \le 3$$



$$V_h \rightsquigarrow \tilde V_h = \tilde V_h^{in} \times \tilde V_h^{ex}, \qquad \text{interpolation on } \tilde V_h: \quad \int_{e_j^k} \tilde I_h v\,ds = \int_{e_j^k} v\,ds$$

② Addition of stabilization terms to the bilinear form $A_h(\cdot,\cdot)$

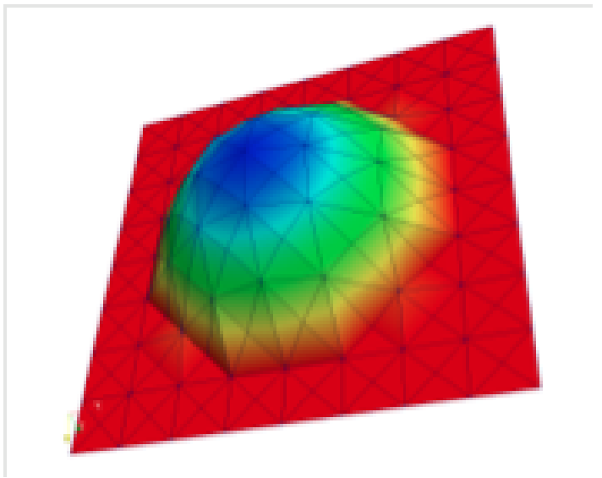
$$\begin{aligned} B_h(u_h, v_h) = & -\sum_{k=in,ex}\sum_{e\in\mathcal{E}_h^k}\int_e \left(\{\eta_k\nabla_n u_h^k\}[v_h^k] + \{\eta_k\nabla_n v_h^k\}[u_h^k]\right)\,ds \\ & + \sum_{k=in,ex}\gamma_k\sum_{e\in\mathcal{E}_h^k}\frac{\eta_k|e|}{|T^{k,l}| + |T^{k,r}|}\int_e [\pi_0 u_h^k]\,[\pi_0 v_h^k]\,ds \end{aligned}$$



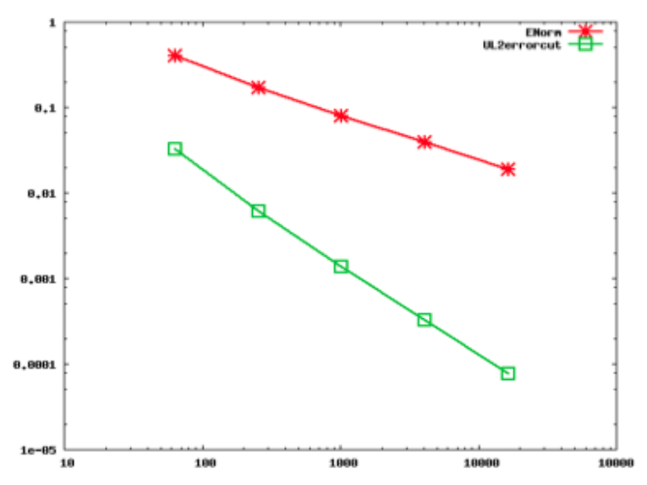
- Obtained results for elliptic and Stokes equations:
  - Existence and uniqueness of the discrete solution
  - Interpolation and a priori error estimates
    (optimal and robust w.r.t. geometry and coefficients)
  - Convergence of the second method towards the first one when $\gamma_k \longrightarrow +\infty$

4. Implementation in Concha library and numerical tests

- Implementation of the second method for elliptic equations
- Reference test (Hansbo & Hansbo '02)
  - exact solution with highly discontinuous coefficients
  - optimal convergence rates
  - similar results to conforming f.e.



(a) Computed solution



(b) L^2 and energy errors

5. Ongoing and future works

- Ongoing works:
  - Implementation in the library Concha (first method, Stokes equations ...)
  - Development of NXFEM for non-standard interface conditions
  - Paper on NXFEM with nonconforming f.e.
- Future works:
  - Giesekus model for a thin membrane (development and implementation)
  - Validation for realistic test-cases (two-phase flow, red blood cell...)
  - Paper on asymptotic modeling of membrane

6. Publications and conferences

- D. Capatina, R. Luce, H. El-Otmany, N.Barrau: *NXFEM for solving non-standard transmission problems*, NM2PorousMedia, Dubrovnik, September 29th - October 3rd 2014
- D. Capatina, S. Delage-Santacreu, H. El-Otmany, D. Graebling: *Robust NXFEM method for a nonconforming approximation on an elliptic problem*, XIth World Congress on Computational Mechanics, Barcelona, 20th-25th July 2014 *(mini-symposium)*
- D. Capatina, S. Delage-Santacreu, H. El-Otmany, D. Graebling: *Modélisation du comportement de globules rouges dans un écoulement sanguin*, Journées Bordeaux-Pau-Toulouse, Anglet, 19-20 September 2013
- *NXFEM with nonconforming finite elements*, PhD Students Seminar, LMAP, 2012

7. Other research activities

- Participation to the ECCOMAS Conference "XFEM 2013", Lyon (France)
- Elected member in LMAP Council, 2013
- Co-organization of *PhD Students Seminar*, LMAP, 2013-2014
- Co-author of *Guide des doctorants*, Doctoral School 211, University of Pau
- Teaching (64 h/year), 2013-2014
- 8 training courses (9 ECTS)

Days of Doctoral School, June 05-06th, 2014

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