NXSFEM for elliptic problem on random domains

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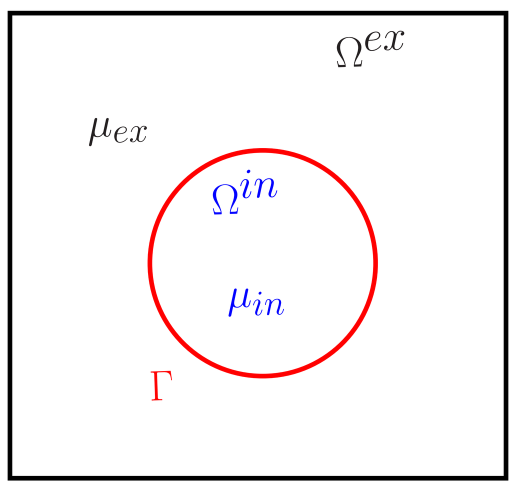
General situation : motivation and goals

The NXFEM (Nitsche Extended Finite Element Method) is designed to take into account discontinuities on non-aligned meshes. This method is well studied on triangular meshes with conforming approximations of the unknowns.
On the one hand, the goals of this study are the theoretical and numerical analysis of the novel method NXSFEM (Nitsche's eXtended Stochastic Finite Element Method) using different types of finite elements (conforming, non-conforming ...) on triangular meshes. This method is based on a mariage between NXFEM method and spectral stochastic method. On the other hand, we propose to apply it to elliptic interface problem on random domains.
In engineering practice, stochastic terms are frequently used and it is therefore important to solve the fluctuations and to dispose of efficient F.E. methods not only to guarantee optimal orders of convergence but also robustness with respect to the geometry of the meshes. The method will be developed in the C++ library Concha.

1. Problem model

We consider and we introduce the associated probability space $(\Xi, \mathcal{B}_\Xi, P)$. Find the solution field u such that it verifies almost surely

$$\begin{cases} -\mathrm{div}(k\nabla u) = f \; in \; \Omega \\ u = 0 \; on \; \partial\Omega \\ [u] = 0 \; on \; \Gamma \\ [k\nabla u \cdot \mathrm{n}] = g \; on \; \Gamma \end{cases} \quad (1)$$



where

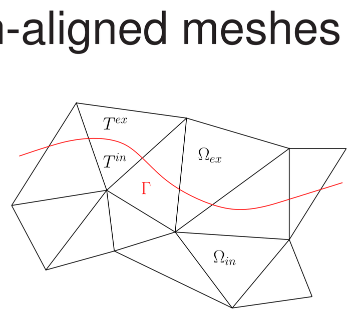
- Ω^{in} and Ω^{ex} are the two random media ($\Omega = \Omega^{in} \cup \Omega^{ex}$)
- $\Omega^i : \xi \in \Xi \longmapsto \Omega^i(\xi) \subset \mathbb{R}^d$ are random variables modeling the geometrical uncertainties.
- Γ is the interface (discontinuity) between Ω^{in} and Ω^{ex}
- u is the unknown variable (for example temperature)
- K is the diffusion coefficient: $k|_{\Omega^{in}} = k^{in}$ and $k|_{\Omega^{ex}} = k^{ex}$.
- f is an external source ($f \in L^2(\Omega)$)
- $[\cdot]$ is jump on discontinuity Γ and $g \in H^{1/2}(\Gamma)$.
- n is the unit outward normal to the boundary.

2. How to solve ?

- For deterministic media and discontinuities aligned with the 2D mesh and standard conditions on the data, we know that the model (1) is well posed.
- There exist efficient numerical methods for solving (1), but there are more suitable when the interface and the mesh don't coincide.
- The NXFEM was introduced by A. and P. Hansbo [3, 2002] on triangular meshes for deterministic media, and is based on a variational formulation with standard finite element spaces, which are locally enriched in such a way that the accurate capturing of an interface not aligned with the mesh is possible.
- This study consist of mariage between NXFEM and spectral stochastic method and compare it to XSFEM for elliptic interface problem.

3. Deterministic case: NXFEM for elliptic interface problem

- Nitsche's Extended Finite Element Method (NXFEM):
  - introduced for conforming approx. of elliptic problems (Hansbo & Hansbo '02)
  - designed to take into account discontinuities on non-aligned meshes by:
    - standard FE spaces enriched on cut cells (doubled d.o.f. : $V_h = V_h^{in} \times V_h^{ex}$)
    - interface conditions treated by Nitsche's method (additional terms in the weak form)



$$a_h(u_h, v_h) = \int_\Omega k\nabla u_h \cdot \nabla v_h \, dx \quad \rightsquigarrow \text{Classical term}$$
$$- \int_\Gamma \{k\nabla_n u_h\} [v_h] \, ds - \int_\Gamma \{k\nabla_n v_h\} [u_h] \, ds \quad \rightsquigarrow \text{Symmetrization}$$
$$+\gamma \int_\Gamma [u_h][v_h] \, dsdP(\xi) \quad \rightsquigarrow \text{Stabilization}$$
$$L_h(v_h) = \int_\Xi \int_{\Omega(\xi)} f v_h dx + \int_\Gamma g v_h ds$$

with γ a stabilization parameter

4. Stochastic case: NXFEM for elliptic interface problem on random media

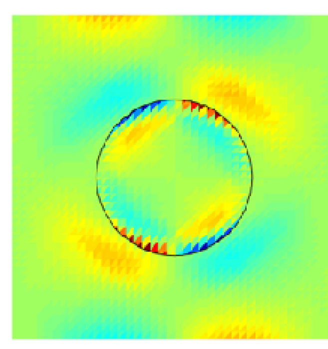
- Development of novel NXSFEM method for:
  - Conforming finite elements
  - Elliptic interface problem
- Variational formula:
  - $a_h(u_h, v_h, \xi) = L_h(v_h, \xi)$ where

$$a_h(u_h, v_h, \xi) = \int_\Xi \int_{\Omega(\xi)} k\nabla u_h \cdot \nabla v_h \, dxdP(\xi) - \int_\Xi \int_{\Gamma(\xi)} \{k\nabla_n u_h\} [v_h] \, dsdP(\xi)$$
$$- \int_\Xi \int_\Gamma \{k\nabla_n v_h\} [u_h] \, dsdP(\xi) + \int_\Xi \int_{\Gamma(\xi)} \gamma_T [u_h][v_h] \, dsdP(\xi)$$
$$L_h(v_h, \xi) = \int_\Xi \int_{\Omega(\xi)} f v_h dxdP(\xi) + \int_\Xi \int_{\Gamma(\xi)} g v_h dsdP(\xi)$$

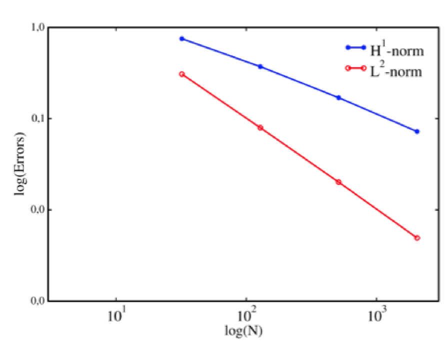
- Main difficulty:
  - Determination of the stabilization parameter γ_T
  - Robustness for geometric inteface and diffusion coefficient
- Proposed solutions
  1. Determination of the mean value in each triangle in order to determine a stabilization parameter γ_T
  2. Adding some weights based on the xpected triangle area
- Obtained results for elliptic interface problem:
  - Existence and uniqueness of the discrete solution almost surely
  - Interpolation and a priori error estimates (optimal and robust w.r.t. geometry and coefficients)

4. Implementation in Concha library and numerical tests

- **Main difficulties:**
  How simulate and evaluate integrals on each part of a random triangle T cut by Γ ?
- **Solution:**
  Built an integration formula on each part of T by using a subdivision into triangles and using Monte Carlo method to evaluate it.
- **Disadvantages:**
  The integration formula depends on the geometry of the random cut triangle T
- Reference test (Nouy & al. '08)
  - exact solution with deterministic highly discontinuous coefficients
  - optimal convergence rates with $\|u\|_{L^2(\Omega)} = \int\int u^2(x, y) dx dP(\omega)$.
  - similar results to conforming f.e.



(a) Computed solution



(b) L^2 and energy errors

6. References

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